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(abstract)

Preconditioning and Fast Summation Techniques for First-Kind Boundary Integral Equations

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1 Introduction

Many problems in potential theory lead to integral equation of the first kind posed on the boundary of a domain in the tree-space. The foremost example is the single layer equation

$$\int_S G(x,y)\sigma(y)\,dS_y = f(x)\,, \qquad x \in S\,. \tag{1}$$

Here f is a given function on the surface S, the function σ is the the unknown and $G(x,y) = 1/4\pi|x-y|$ is the Green's function for the Laplacian.

A standard approach to discretize this integral equation is to use a collocation scheme. For that, partition the surface into a number of pieces (panels) $S = S_1 \cup \ldots \cup S_N$. The numerical solution is the function which is piecewise constant of this subdivision and which satisfies (1) in the centroids x_i of the panels. This leads to the linear system $Aq = f$, where the entries of the matrix A are given by

$$A_{i,j} = \int_{S_j} G(x_i, y)\,dS_y\,.$$

The matrix is non-symmetric, dense and very large for fine discretizations or complex geometries of S. Moreover, since (1) is an integral equation of the first kind, the condition number of A increases with the refinement of the grid.

With the recent development of fast summation techniques it has become possible to reduce the complexity of a matrix-vector product from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$ or even nearly $\mathcal{O}(N)$. Thus the solution of the discretized equations by an iterative scheme like Krylov subspace methods is now feasible.

Among the algorithms which are available for boundary integral equations in complex geometries are wavelet compression [1], particle-mesh techniques [8, 4] the fast multipole method (FMM) [2, 6], and panel clustering [3]. The latter two techniques obtain their efficiency from a hierarchical decomposition of the problem domain. We found this idea also useful for the construction of efficient preconditioners.

2 Preconditioners for Boundary Elements

The basic idea of preconditioning is to approximate the inverse of the system matrix, ideally one would like to have $Ap_i = e_i$ for the columns p_i of the preconditioner P. The principle behind most approaches to construct P has been described by Vavasis [7]: For an index i one selects a small list L of indices which have the largest influence on the i-th component of the unknown vector q. Since the off-diagonal entries of A decay like $1/r$, where r is the distance of two panels, a palpable choice of the list L consists of the mesh neighbors of panel S_i. Then a small system $\bar{A}\bar{p}_i = \bar{e}_i$ is solved for every index i. Here the bars stand for deleting all rows and columns except for those in L. Finally the $\bar{p}_i$'s are expanded back into the i-th row of P by setting entries to zero which correspond to indices in the complement of L .

Several variations of the above approach are possible. For instance, the local inversion preconditioner [5] places the surface into a cube which is subdivided into a number of sub-cubes. Then the preconditioner is constructed block-wise rather than index-wise: Let i denote the list of indices belonging to panels in a sub-cube. Then the list L is chosen to be the indices of panels in the cube and its nearest neighbors. To obtain the i-th block-row of the preconditioner, the matrix $\bar{A}$ is inverted and the columns of the inverse belonging to the list i are expanded into the i-rows of P.

The problem with the overlapping preconditioner is the trade-off between its quality and the size of the lowest-level cubes. To ensure a bounded condition number of the preconditioned system as the discretization is refined, the size of the cubes must be kept constant. However, this is prohibitive because the cost to construct the preconditioner would be $\mathcal{O}(N^3)$. On the other hand, if the size of the cubes is chosen to contain a fixed number of panels as the grid is refined, the efficiency of the preconditioner will deteriorate.

3 Hierarchical Approach

The idea behind the overlapping preconditioner is to neglect interactions of far away panels. This may not be appropriate for the operator in (1), because the $1/r$ decay of the kernel cancels with the $\mathcal{O}(r)$ increase of panels at distance r.

To account for this effect we take the following approach: First, partition the cube containing S into a hierarchy of sub-cubes. That is, the top level cube is the cube itself, and the cubes in level l are uniform refinements the level $l-1$ cubes into eight sub-cubes.

Instead of neglecting the far field in the generation of the preconditioner, assume that distant interactions may be approximated by charge distributions which are constant within cubes at certain levels. The more distant the interaction, the higher the level. This results in a linear system for each non-empty bottom level cube. The matrices are then inverted and the columns corresponding to the cube expanded into the appropriate rows of the preconditioner.

If the bottom level cubes are chosen to contain at most a fixed number of

panels, then the size of one linear system can be shown to be $\mathcal{O}(\log N)$. Since there are $\mathcal{O}(N)$ systems to solve, the total complexity for the construction is $\mathcal{O}(N \log N)$.

The preconditioner is not sparse, but its entries are constant for rows and columns belonging to certain pairs of cubes in the hierarchy. This property makes fast matrix-vector multiplication algorithms possible.

In the talk will present results which demonstrate the improvements obtained by using hierarchical preconditioners rather than local inversion preconditioners for complex geometries which arise in a variety of engineering domains.

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